

Resolvable h -sun designs

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Abstract

In this article we completely determine the spectrum for uniformly resolvable decompositions of the complete graph K_v into r 1-factors and s classes containing only copies of h -suns.

*Supported by PRIN and I.N.D.A.M (G.N.S.A.G.A.), Italy

†Supported by PRIN, PRA and I.N.D.A.M (G.N.S.A.G.A.), Italy

‡Supported by MIUR and by C. N. R. (G. N. S. A. G. A.), Italy

§Supported by PRIN, PRA and I.N.D.A.M (G.N.S.A.G.A.), Italy

AMS Subject classification: 05B05.

Keywords: Resolvable graph decomposition; uniform resolutions; h -sun designs.

1 Introduction

Given a collection \mathcal{H} of graphs, an \mathcal{H} -decomposition of a graph G is a decomposition of the edge set of G into subgraphs (called *blocks*) isomorphic to some element of \mathcal{H} . Such a decomposition is said to be *resolvable* if it is possible to partition the blocks into classes \mathcal{P}_i (often referred to as *parallel classes*) such that every vertex of G appears in exactly one block of each \mathcal{P}_i ; a class is called *uniform* if every block of the class is isomorphic to the same graph from \mathcal{H} . A resolvable \mathcal{H} -decomposition of G is sometimes also referred to as an \mathcal{H} -factorization of G , and a class can be called an \mathcal{H} -factor of G . The case where $\mathcal{H} = \{K_2\}$ (a single edge) is known as a *1-factorization*; for $G = K_v$ it is well known to exist if and only if v is even. A single class of a 1-factorization, that is a pairing of all vertices, is also known as a *1-factor* or *perfect matching*.

Uniformly resolvable decompositions of K_v have been studied in [3], [7], [10], [12], [15], [16], [17], [19] and [20]. Moreover when $\mathcal{H} = \{G_1, G_2\}$ the question of the existence of a uniformly resolvable decomposition of K_v into $r > 0$ classes of G_1 and $s > 0$ classes of G_2 have been studied in the case in which the number s of G_2 -factors is maximum. Rees and Stinson [18] have solved the case $\mathcal{H} = \{K_2, K_3\}$; Hoffman and Schellenberg [9] the case $\mathcal{H} = \{K_2, C_k\}$; Dinitz, Ling and Danziger [4] the case $\mathcal{H} = \{K_2, K_4\}$; Küçükçifçi, Milici and Tuza [10] the case $\mathcal{H} = \{K_3, K_{1,3}\}$; Küçükçifçi, Lo Faro, S. Milici and Tripodi [11] the case $\mathcal{H} = \{K_2, K_{1,3}\}$.

1.1 Definitions and notation

An h -sun ($h \geq 3$) is a graph with $2h$ vertices $\{a_1, a_2, \dots, a_h, b_1, b_2, \dots, b_h\}$, consisting of an h -cycle $C_h = (a_1, a_2, \dots, a_h)$ and a 1-factor $\{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_h, b_h\}\}$; in what follows we will denote the h -sun by $S(C_h) = (a_1, a_2, \dots, a_h; b_1, b_2, \dots, b_h)$ or $S(C_h)$. An h -sun is also called a crown graph [8]. The spectrum problem for a h -sun system of order v have been solved for $h = 3, 4, 5, 6, 8$ [5, 13, 14]. Moreover cyclic h -sun systems of order v have been studied in [5, 6, 23].

Let $C_{m(n)}$ denote the graph with vertex set $\bigcup_{i=1}^m X^i$, with $|X^i| = n$ for $i = 1, 2, \dots, m$ and $X^i \cap X^j = \emptyset$ for $i \neq j$, and edge set $\{\{u, v\} : u \in X^i, v \in X^j, |i - j| \equiv 1 \pmod{m}\}$.

In this paper we study the existence of a uniformly resolvable decomposition of K_v having r 1-factors and s classes containing only h -suns; we will use the notation $(K_2, S(C_h))$ -URD($v; r, s$) for such a uniformly resolvable decomposition of K_v . Further, we will use the notation $(K_2, S(C_h))$ -URGDD(r, s) of

$C_{m(n)}$ to denote a uniformly resolvable decomposition of $C_{m(n)}$ into r 1-factors and s classes containing only h -suns.

2 Necessary conditions

In this section we will give necessary conditions for the existence of a uniformly resolvable decomposition of K_v into r 1-factors and s classes of h -suns.

Lemma 2.1. *If there exists a $(K_2, S(C_h))$ -URD($v; r, s$), $s > 0$, then $v \equiv 0 \pmod{2h}$ and $s \equiv 0 \pmod{2}$.*

Proof. Assume that there exists a $(K_2, S(C_h))$ -URD($v; r, s$), $s > 0$. By resolvability it follows that $v \equiv 0 \pmod{2h}$. Counting the edges of K_v we obtain

$$\frac{rv}{2} + \frac{(2h)sv}{2h} = \frac{v(v-1)}{2}$$

and hence

$$r + 2s = (v-1). \quad (1)$$

Denote by R the set of r 1-factors and by S the set of s parallel classes of h -suns. Since the classes of R are regular of degree 1, we have that every vertex x of K_v is incident with r edges in R and $(v-1) - r$ edges in S . Assume that the vertex x appears in a classes with degree 3 and in b classes with degree 1 in S . Since

$$a + b = s \quad \text{and} \quad 3a + b = v - 1 - r,$$

the equality (1) implies that

$$3a + b = 2(a + b) \Rightarrow a = b$$

and hence $s = 2a$. This completes the proof. \square

Given $v \equiv 0 \pmod{2h}$, $h \geq 3$, define $J(v)$ according to the following table:

v	$J(v)$
$0 \pmod{4h}$	$\{(3 + 4x, \frac{v-4}{2} - 2x), x = 0, 1, \dots, \frac{v-4}{4}\}$
$2h \pmod{4h}$, h even,	$\{(3 + 4x, \frac{v-4}{2} - 2x), x = 0, 1, \dots, \frac{v-4}{4}\}$
$2h \pmod{4h}$, h odd,	$\{(1 + 4x, \frac{v-2}{2} - 2x), x = 0, 1, \dots, \frac{v-2}{4}\}$

Table 2: The set $J(v)$.

Since a $(K_2, S(C_h))$ -URD($v; v-1, 0$) exists for every $v \equiv 0 \pmod{2}$, we focus on $v \equiv 0 \pmod{2h}$, $h \geq 3$.

Lemma 2.2. *If there exists a $(K_2, S(C_h))$ -URD($v; r, s$) then $(r, s) \in J(v)$.*

Proof. Assume there exists a $(K_2, S(C_h))$ -URD($v; r, s$). Lemma 2.1 and Equation (1) give $s \equiv 0 \pmod{2}$ and $r \equiv (v-1) \pmod{4}$ and so

- if $v \equiv 0 \pmod{4h}$, then $r \equiv 3 \pmod{4}$,
- if $v \equiv 2h \pmod{4h}$, h even, then $r \equiv 3 \pmod{4}$,
- if $v \equiv 2h \pmod{4h}$, h odd, then $r \equiv 1 \pmod{4}$.

Letting $r = a + 4x$, $a = 1$ or 3 , in Equation (1), we obtain $2s = (v-1) - a - 4x$; since s cannot be negative, and x is an integer, the value of x has to be in the range as given in the definition of $J(v)$. \square

Let now $URD(v; K_2, S(C_h)) := \{(r, s) : \exists (K_2, S(C_h))\text{-URD}(v; r, s)\}$. In this paper we completely solve the spectrum problem for such systems, i.e., characterize the existence of uniformly resolvable decompositions of K_v into r 1-factors and s classes of h -suns by proving the following result:

Main Theorem. *For every $v \equiv 0 \pmod{2h}$, $URD(v; K_2, S(C_h)) = J(v)$.*

3 Small cases and basic lemmas

Lemma 3.1. $URD(6; K_2, S(C_3)) \supseteq J(v)$.

Proof. The case $(5, 0)$ is trivial. For the case $(1, 2)$, let $V(K_6) = \mathbb{Z}_6$ and the classes listed below:

$\{(0, 1, 2; 5, 4, 3)\}, \{(3, 5, 4; 0, 1, 2)\}, \{\{0, 4\}, \{1, 3\}, \{2, 5\}\}.$ \square

Lemma 3.2. $URD(12; K_2, S(C_3)) \supseteq J(v)$.

Proof. The case $(11, 0)$ is trivial. For the remaining cases, let $V(K_{12}) = \mathbb{Z}_{12}$ and the classes listed below:

- $(3, 4)$:
 $\{(0, 4, 8; 10, 2, 7), (1, 5, 9; 11, 3, 6)\}, \{(2, 6, 10; 8, 0, 5), (3, 7, 11; 9, 1, 4)\},$
 $\{(0, 5, 11; 9, 2, 6), (1, 4, 10; 8, 3, 7)\}, \{(2, 7, 9; 11, 0, 4), (3, 6, 8; 10, 1, 5)\},$
 $\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}\},$
 $\{\{0, 2\}, \{1, 3\}, \{4, 7\}, \{5, 6\}, \{8, 11\}\}, \{9, 10\},$
 $\{\{0, 3\}, \{1, 2\}, \{4, 6\}, \{5, 7\}, \{8, 10\}, \{9, 11\}\};$
- $(7, 2)$:
 $\{(0, 4, 8; 10, 2, 7), (1, 5, 9; 11, 3, 6)\}, \{(2, 6, 10; 8, 0, 5), (3, 7, 11; 9, 1, 4)\},$
 $\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}, \{10, 11\}\},$
 $\{\{0, 2\}, \{1, 3\}, \{4, 7\}, \{5, 6\}, \{8, 11\}\}, \{9, 10\},$

$$\begin{aligned}
& \{\{0, 3\}, \{1, 2\}, \{4, 6\}, \{5, 7\}, \{8, 10\}, \{9, 11\}\}, \\
& \{\{0, 5\}, \{1, 10\}, \{2, 11\}, \{3, 4\}, \{6, 8\}, \{7, 9\}\}, \\
& \{\{0, 7\}, \{1, 8\}, \{2, 5\}, \{3, 10\}, \{4, 9\}, \{6, 11\}\}, \\
& \{\{0, 9\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 10\}, \{5, 11\}\}, \\
& \{\{0, 11\}, \{1, 4\}, \{2, 9\}, \{3, 6\}, \{5, 8\}, \{7, 10\}\}.
\end{aligned}$$

□

Lemma 3.3. *There exists a $(K_2, S(C_h))$ -URGDD(r, s) of $C_{h(2)}$, for every $(r, s) \in \{(0, 2), (4, 0)\}$.*

Proof. Consider the sets $X^i = \{a_i, b_i\}$, for $i = 1, 2, \dots, h$, and take the classes listed below (where we assume $h + 1 = 1$):

- $(0, 2)$:
 $\{(a_1, a_2, \dots, a_h; b_2, b_3, \dots, b_h, b_1)\}, \{(b_1, b_2, \dots, b_h; a_2, a_3, \dots, a_h, a_1)\};$
- $(4, 0)$, h even:
 $\{\{a_{1+2i}, a_{2+2i}\}, \{b_{1+2i}, b_{2+2i}\} : i = 0, 1, \dots, \frac{h}{2} - 1\},$
 $\{\{a_{2+2i}, a_{3+2i}\}, \{b_{2+2i}, b_{3+2i}\} : i = 0, 1, \dots, \frac{h}{2} - 1\},$
 $\{\{a_{1+i}, b_{2+i}\} : i = 0, 1, \dots, h - 1\},$
 $\{\{a_{2+i}, b_{1+i}\} : i = 0, 1, \dots, h - 1\};$
- $(4, 0)$, h odd:
 $\{\{a_{1+2i}, a_{2+2i}\}, \{b_{2+2i}, b_{3+2i}\} : i = 0, 1, \dots, \frac{h-5}{2}\} \cup \{\{a_{h-2}, a_{h-1}\}, \{a_h, b_{h-1}\}, \{b_1, b_h\}\},$
 $\{\{a_{2+2i}, a_{3+2i}\}, \{b_{1+2i}, b_{2+2i}\} : i = 0, 1, \dots, \frac{h-3}{2}\} \cup \{\{a_1, b_h\}\},$
 $\{\{a_{2+i}, b_{1+i}\} : i = 0, 1, \dots, h - 3\} \cup \{\{a_1, a_h\}, \{b_{h-1}, b_h\}\},$
 $\{\{a_{1+i}, b_{2+i}\} : i = 0, 1, \dots, h - 1\}.$

□

4 Main results

Lemma 4.1. *For every $v \equiv 0 \pmod{4h}$, $h \geq 3$, $URD(v; K_2, S(C_h)) \supseteq J(v)$.*

Proof. Let $v = 4ht$. The case $h = 3$ and $t = 1$ corresponds to a $(K_2, S(C_3))$ -URD $(12; r, s)$ which follows by Lemma 3.2. For $t \geq 2$, start with a C_h -factorization $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_l$, $l = ht - 1$, of $K_{2ht} - F$ which comes from [9] and give weight 2 to each point of X . Fixed any integer $0 \leq x \leq l$, for each h -cycle C of x parallel classes place on $C \times \{1, 2\}$ a copy of a $(K_2, S(C_h))$ -URGDD(4, 0) of $C_{h(2)}$, while for each h -cycle C of the remaining classes place a copy of a $(K_2, S(C_h))$ -URGDD(0, 2) of $C_{h(2)}$ (the input designs are from Lemma 3.3); for each edge $e \in F$ consider a 1-factorization of K_4 on $e \times \{1, 2\}$. The result is a resolvable decomposition of K_v into $3 + 4x$ 1-factors and $\frac{v-4}{2} - 2x$ classes of h -suns.

□

Lemma 4.2. *For every $v \equiv 2h \pmod{4h}$, $h \geq 3$ even, $URD(v; K_2, S(C_h)) \supseteq J(v)$.*

Proof. Let $v = 2h + 4ht$. Starting with a C_h -factorization of $K_{h+2ht} - F$, which comes from [9], the assertion follows by a similar argument as in Lemma 4.1. \square

Lemma 4.3. *For every $v \equiv 2h \pmod{4h}$, $h \geq 3$ odd, $URD(v; K_2, S(C_h)) \supseteq J(v)$.*

Proof. Let $v = 2h + 4ht$. The case $h = 3$ and $t = 0$ corresponds to a $(K_2, S(C_3))$ -URD $(6; 1, 2)$ which follows by Lemma 3.1. For $t \geq 1$, start with a C_h -factorization $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_l$, $l = ht + \frac{h-1}{2}$, of K_{h+2ht} which comes from [1] and give weight 2 to each point of X . Fixed an integer $0 \leq x \leq l$, for each h -cycle C of x parallel classes place on $C \times \{1, 2\}$ a copy of a $(K_2, S(C_h))$ -URGDD $(4, 0)$ of $C_{(h)2}$, while for each h -cycle C of the remaining classes place a copy of a $(K_2, S(C_h))$ -URGDD $(0, 2)$ of $C_{(h)2}$ (the input designs are from Lemma 3.3). If we consider also the 1-factor consisting of the edges $\{x_1, x_2\}$ for $x \in X$, the result is a resolvable decomposition of K_v into $1 + 4x$ 1-factors and $\frac{v-2}{2} - 2x$ classes of h -suns. \square

Combining Lemmas 4.1, 4.2, 4.3, we obtain our main theorem.

Theorem 4.4. *For each $v \equiv 0 \pmod{2h}$, $URD(v; K_2, S(C_h)) = J(v)$.*

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